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Linear Fractional Recurrences

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We let $\mathbf{C}^k$ denote complex Euclidean space, and we consider birational maps $f : \mathbf{C}^k \dashrightarrow \mathbf{C}^k$ of the form

$$f = f_{\alpha,\beta}(x_1, \ldots, x_k) = \left(x_2, \ldots, x_k, \frac{\alpha \cdot x}{\beta \cdot x}\right) \tag{1}$$

where $\alpha \cdot x = \sum \alpha_j x_j$ and $\beta \cdot x = \sum \beta_j x_j$. One feature of these maps is that they seem to be the simplest possible nonlinear maps. Something which has interested us is the question of periodicities: *What are the constants* $\alpha = (\alpha_1, \ldots, \alpha_k)$ *and* $\beta = (\beta_1, \ldots, \beta_k)$ *for which* $f_{\alpha,\beta}$ *is periodic?* By *periodic* we mean that $f^N = f \circ \cdots \circ f$ is the identity map for some N. We refer to [GL] and [KL] for further discussion. This question remains unsolved in general, but there is one observation we have made with Kyounghee Kim (see [BK2]):

Theorem 1. *If* $a = (-1)^{1/k}$ *and*

$$\beta = (a^{k-1}, 1, 0, \ldots, 0) \text{ and } \alpha = (a^{k-2}/(1-a), 0, a^{k-2}, \ldots, a^2, a, 1) \tag{2}$$

then $f_{\alpha,\beta}$ *is periodic with period* $4k$.

Thus for each k, there are k different maps of the form (1) which have period $4k$. We remark that the proof given in [BK2] is somewhat indirect. Namely, we consider $f_{\alpha,\beta}$ as a map of $\mathbf{P}^k$. Then we construct a blowup space $\pi : X \to \mathbf{P}^k$ and study the induced map $f_X := \pi^{-1} \circ f \circ \pi : X \dashrightarrow X$. We then determine the induced map f_X^* on $H^{1,1}(X)$. We show that the eigenvalues of f_X^* are roots of unity and that f_X^* has period $4k$. After this, we show that f^{4k} is the identity.

Let us recall the situation for dimension 2 (see [BK1]):

Theorem 2. *If* $k = 2$, *then the only possible (nontrivial) periods for maps (1) are 6, 5, 8, 12, 18, and 30.*

In this case, it is possible to enumerate all the possible values of α and β and to verify directly that specific examples have the stated periods. The more difficult issue is to show that these are the *only* periodic possibilities.

We also consider the case of dimension 3 (see [BK3]):

Theorem 3. *If* $k = 3$, *then the only possible (nontrivial) periods for maps (1) are 8 and 12.*

The maps of period 12 which arise in Theorem 3 correspond to the maps in the case $k = 3$ in Theorem 1. The period 8 maps are given by:

$$f(x) = \left(x_2, x_3, \frac{1 + x_2 + x_3}{x_1}\right), \qquad f(x) = \left(x_2, x_3, \frac{-1 - x_2 + x_3}{x_1}\right)$$

We note that the maps that had been observed earlier were the ones of period 8. The first of these was found by Lyness [Ly], and the second one is due to Csörnyei and Laczkovic [CL]. The behavior of the maps (1) is more complicated in dimension 3 than it was in dimension 2. One explanation for this is that the difficulties arise from blow-down and blow-up behaviors. In dimension 2, all such behavior is either a curve blowing down to a point or a point blowing up to a curve. In dimension 3, a hypersurface can blow down either to a curve or to a point, and vice versa. Further, there can be blow-up behavior without blow-down behavior. For instance, we can have a birational map $g : X \dashrightarrow Y$ and curves $\mathcal{C} \subset X$ and $\mathcal{C}' \subset Y$ such that $g : X - \mathcal{C} \to Y - \mathcal{C}'$ is a biholomorphism, but each point of $\mathcal{C}$ blows up to $\mathcal{C}'$. The difference with dimension 2 is that the Jacobian of g is nonsingular (invertible) at each point of $X - \mathcal{C}$.

We know little about the case $k \geq 4$. In particular, we do not know whether there are nontrivial periods other than the ones given by the maps in Theorem 1 when $k \geq 4$.

One feature that has attracted us to the maps (1) is that the are in some sense the simplest nonlinear maps. Both f and its inverse have degree 2. That is, on $\mathbf{P}^k$, the maps (1), as well as their inverses, are both written in terms of homogeneous polynomials of degree 2. In general, however, when $k = 3$ the inverse of a quadratic map can have degree 2, 3, or 4. The degree of a mapping, however, is not invariant under birational conjugacy. That is, if L is linear (and thus of degree 1), and if φ is birational, then $\varphi^{-1} \circ L \circ \varphi$ can be nonlinear and have degree higher than one. We now define the dynamical degree, which is more natural as a dynamical invariant.

If $f : X \dashrightarrow Y$ is a rational map, then there is a well-defined pullback on cohomology $f^* : H^{p,q}(Y) \to H^{p,q}(X)$ (see [G]). Using this, we may define the dynamical degrees as follows. We then define the ℓ-th dynamical degree as

$$\delta_\ell(f) := \lim_{n\to\infty} ||(f^n)^*|_{H^{\ell,\ell}(X)}||^{1/n} \tag{3}$$

Thus $\delta_\ell(f)$ measures the exponential rate of growth of f on $H^{\ell,\ell}(X)$, which, loosely speaking, corresponds to objects of codimension 2ℓ. δ_k corresponds to the topological (mapping) degree of f. If $X = \mathbf{P}^k$, then $H^{1,1}(\mathbf{P}^k, \mathbf{Z}) \cong \mathbf{Z}$, and $f^*|_{H^{1,1}(\mathbf{P}^k)} = deg$, where deg denotes the usual degree in the representation of f in terms of homogeneous polynomials. That is, if $H = \{\sum c_j x_j = 0\}$ is the class of a hyperplane in $\mathbf{P}^k$, then $f^*H = \{\sum c_j f_j = 0\} = (deg)H$. δ_k corresponds to the topological (mapping) degree of f. The dynamical degree is an important measure of complexity for a rational dynamical systerm, and the quantity $\delta_\ell(f)$ was shown to be an invariant of birational conjugacy by Dinh and Sibony [DS].

We note that our search for periodicities in the family (1) is essentially a process of eliminating the non-periodic maps. Our original approach was to find the α and β for which $\delta_1(f_{\alpha,\beta}) > 1$. Obviously, if the degree growth is exponential, then the map is not periodic. With this approach, our study of the maps (1) quickly becomes an analysis of the critical maps; we will say that $f_{\alpha,\beta}$ is *critical* if $\beta_2 = \beta_3 = 0$ and $\beta_1\alpha_2\alpha_3 \neq 0$.

Theorem 4. *For a generic critical map, the first dynamical degree $\delta_1(f_{\alpha,\beta}) \sim 1.32472$, the largest root of $x^3 - x - 1$.*

For $1 < \ell < k$, the dynamical degree δ_ℓ is not well understood. Of course, if f is in fact holomorphic, then δ_ℓ is the spectral radius of the map $f^*|_{H^{\ell,\ell}(X)}$. However, when f is not holomorphic, a class $\eta \in H^{\ell,\ell}$ might be carried by a cycle inside the indeterminacy locus, so the interpretation of $f^*\eta$ is not obviously gotten by pulling back the cycle defining η.

In the case of dimension 3, we have the Poincaré duality $\langle \cdot, \cdot \rangle$ between $H^{1,1}(X)$ and $H^{2,2}(X)$ and thus an adjoint f_* acting on $H^{2,2}$. That is, for $\xi \in H^{1,1}(X)$ and $\eta \in H^{2,2}(X)$, we have $\langle f^*\eta, \xi\rangle = \langle \eta, f_*\xi\rangle$. Since f is birational, we also have the pullback of $f^{-1} : X \dashrightarrow X$ acting on $H^{1,1}(X)$. Thus the pullback $(f^{-1})^*|_{H^{1,1}}$ is equivalent under this duality to $f^*|_{H^{2,2}}$. This gives us that $\delta_2(f) = \delta_1(f^{-1})$.

This leads to the question whether there is any family of rational maps for which it is possible to determine δ_ℓ for $1 < \ell < k$. At present, the only general family for which δ_ℓ is known is the family of monomial maps. That is, we let $A = (a_{i,j})$ be a $k \times k$ matrix with integer entries. (The interesting case here is when A contains negative entries.) Then we define a rational map $g_A : \mathbf{C}^k \dashrightarrow \mathbf{C}^k$ by setting

$$g_A(x_1, \ldots, x_k) = \left(\prod_j x_j^{a_{1,j}}, \ldots, \prod_j x_j^{a_{k,j}} \right) \tag{4}$$

which, heuristically, is $g_A = e^{A \log x}$. A basic property is that iteration of the monomial map corresponds to matrix multiplication: $(g_A)^n = g_{A^n}$. As we noted above, δ_ℓ is a birational invariant, so we can choose our space to work on. We choose to work on the manifold $X = (\mathbf{P}^1)^k$, which is the compactification of $\mathbf{C}^k$ obtained by taking the product of the compactifications of the factors $\mathbf{C}$. It is evident that a basis for $H^{1,1}(X)$ is given by the coordinate hyperplanes $\{x_j = 0\}$. Further, a basis of $H^{p,p}(X)$ is given by $\{x_{i_1} = \cdots = x_{i_p} = 0\}$, where $1 \le i_1 < \cdots < i_p \le k$ consists of p distinct indices. We also consider the following matrix operation: Given a matrix $M = (m_{i,j})$, we define $|M| = (|m_{i,j}|)$ to be the matrix obtained by taking absolute values of all the entries. J-L Lin [Li] has shown that g_A^* is given by the exterior product of A:

Proposition. *With respect to this basis, $g_A^*|_{H^{p,p}}$ is given by $|\bigwedge^p A|$.*

Working from this Proposition, Lin [Li] obtained the following result, which was also obtained independently using different methods by Favre and Wulcan [FW]:

Theorem 5. *If g_A is as in (4), then $\delta_p(g_A) = |\mu_1 \cdots \mu_p|$, where $|\mu_1| \ge |\mu_2| \ge \cdots \ge |\mu_p|$ are the eigenvalues of A.*

The family (1) also leads us to automorphisms. To say f *is an automorphism* means that there is a blowup space $\pi : X \to \mathbf{P}^k$ (perhaps involving iterated blowups) such that the induced map $f_X := \pi^{-1} \circ f \circ \pi$ is an automorphism of X. De Fernex and Ein [dFE] have shown that if a map is periodic (in any dimension), then it is an automorphism in the sense above (see [dFE]).

[BK1] showed that looking inside the 2-dimensional version of family (1) reveals a number of rational surface automorphisms with positive entropy. When we go to higher dimension, we must be more careful. For a general manifold X of dimension k, we follow Dolgachev and Ortland [DO] and say that $f : X \dashrightarrow X$ is a *pseudo-automorphism* if f and f^{-1} are local diffeomorphisms at all points away from the indeterminacy locus. In dimension 2, f is a pseudo-automorphism if and only if it is an automorphism, but not in higher dimension. In [BK3] we find that the family (1) contains pseudo-automorphisms of positive entropy on spaces which are blowups of $\mathbf{P}^3$:

Theorem 6. *Suppose that $\alpha = (a, 0, \omega, 1)$ and $\beta = (0, 1, 0, 0)$ where $a \in \mathbf{C} \setminus \{0\}$ and ω is a non-real cube root of the unity. Then there is a modification $\pi : Z \to \mathbf{P}^3$ such that f_Z is a pseudo-automorphism. The dynamical degrees $\delta_1(f) = \delta_2(f) \cong 1.28064 > 1$ are equal and are given by the largest root of $t^8 - t^5 - t^4 - t^3 + 1$. The entropy of f_Z is the logarithm of the dynamical degree and is thus positive.*

In addition, there is a sort of integrability for these maps:

Theorem 7. *For the mappings in Theorem 1, there is a 1-parameter family of surfaces $S_c \subset Z$, $c \in \mathbf{C}$ which have the invariance $fS_c = S_{\omega c}$. For generic c, S_c is $K3$, and the restriction $f^3|_{S_c}$ is an automorphism. For generic c and c', the surfaces S_c and $S_{c'}$ are biholomorphically inequivalent, and the automorphisms $f^3|_{S_c}$ and $f^3|_{S_{c'}}$ are not smoothly conjugate.*

The surface S_0 is invariant, and the restriction f_{S_0} is an automorphism which has the same entropy as f. This is smaller than the entropy of the automorphism constructed in [M, Theorem 1.2] and is thus the smallest known entropy for a *projective* $K3$ surface automorphism.

Let us write $f_c := f|_{S_c}$ for the restriction to S_c. The automorphisms of $K3$ surfaces were studied by Cantat [C]. In our case, it follows that there are positive, closed currents $\mu_c^\pm$ on S_c such that $f_c^{3*}\mu_c^\pm = \delta^{\pm 3}\mu_c^\pm$, and $\mu_c := \mu_c^+ \wedge \mu_c^-$ is the unique measure of maximal entropy.

We let $\alpha^+ \in H^{1,1}(Z)$ denote the class which is expanded by f_Z^*. If α^+ is nef, then by Diller and Guedj [DG] there is an invariant current T^+ in α^+ which is invariant (expanded by f_Z^*) and which has the "attractor" property that for all smooth currents Ξ^+ in the class of α^+, the normalized pullbacks $\delta^{-n} f_Z^{*n}\Xi^+ \to T^+$. Inspired by Bayraktar [B], we can construct Z such that α^+ to be nef. Similarly, we have a corresponding current T_Z^-, and we may wedge these two currents to obtain an invariant (2,2)-current $T := T^+ \wedge T^-$, which satisfies $f^*T = T$. These currents have properties analogous to the bifurcation currents studied by Dujardin and Favre [DuF]. That is, their slices by the invariant $K3$ surfaces give the corresponding invariant currents/measures for (f_c, S_c): $T^+|_{S_c} = \mu_c^+$, and $T|_{S_c} = \mu_c$.

The following mappings have quadratic degree growth and complete integrability:

Theorem 8. *Suppose that $\beta = (0,1,0,0)$ and either $\alpha = (0,0,\omega,1)$ or $\alpha = (a,0,1,1)$ where $a \in \mathbf{C} \setminus \{1\}$, $\omega \neq 1$, and $\omega^3 = 1$. Then the degree of f^n grows quadratically in n. Further, there is a modification $\pi : Z \to \mathbf{P}^3$ such that f_Z is a pseudo-automorphism. There is a two-parameter family of surfaces S_c, $c = (c_1, c_2) \in \mathbf{C}^2$ which are invariant under f^3. For generic c and c', S_c is a smooth $K3$ surface, and $S_c \cap S_{c'}$ is a smooth elliptic curve.*

For the mappings in Theorems 4 and 8, f is reversible on the level of cohomology: f_Z^* is conjugate to $(f_Z^{-1})^* = (f_Z^*)^{-1}$. The identity $\delta_1(f) = \delta_2(f)$ for such maps is a consequence of the duality between $H^{1,1}$ and $H^{2,2}$, so they are not cohomologically hyperbolic, in the terminology of [G]. If any of the maps of Theorems 6 and 8 acts on $\mathbf{P}^3$, then it is evident that the variety $\mathcal{R}_0 = \{x_0x_1x_2x_3 = 0\}$ is invariant. After the blow-up $\pi : Z \to \mathbf{P}^3$, we have a divisor $\mathcal{R} := \pi^{-1}\mathcal{R}_0$, which now contains 8 components. In fact, $\mathcal{R}$ is an invariant 8-cycle of surfaces under f_Z. The family of invariant $K3$ surfaces degenerates and becomes singular at a $\mathcal{R}$. We have seen that f_Z is a pseudo-automorphism and not an automorphism. This is a property of f and not, somehow, a defect of our choice of a particular blowup space Z. In [BK3] we showed:

Theorem 7. *Let f be a map from Theorems 1 and 3. If $a \neq 1$, then the restriction $f^8|_{\{x_3=0\}}$ is not birationally equivalent to a surface automorphism. Thus there is no proper modification $\pi : W \to \mathbf{P}^3$ such that the induced map f_W is an automorphism.*

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